Maxwell-Boltzmann and Fermi-Dirac Statistical Mechanics Basics

Distribution Functions

For f(*something*) that describes the distribution of a particle quantity over something (like the number of particles distributed over energy or location over a span of space), then

f(*something*) d*something* = Probability of finding a particle between *something* and *something* + d*something*

f(x) dx = Probability of finding a particle with a location between x and x + dx

f(E) dE = Probability of finding a particle with energy between E and E + dE

The expectation value of *something* described by this distribution is

$$\langle \mathit{something} \rangle = \int_{-\infty}^{\infty} \mathit{something}\ f(\mathit{something})\ d\mathit{something}$$

Energy Distributions of Particles

To describe how the energy is distributed among particles in large collections (gases, liquids and solids), physicists developed different energy distributions based on the types of particles

$$n(E)dE = g(E)F_{\text{smart guys}}$$

smart guy factor expressing behavior of particle type: classical, Fermion, or Boson

density of states (number of E < available energy states < E + dE

number of particles with E < energy < E + dE [n(E) = number/energy …number energy density]

Types of Statistics:

Classical Particles: Maxwell-Boltzmann Statistics

- Particles widely separated
- Particles interact only via totally elastic collisions
- Particles can have any energy

Quantum Particles

- Particles Described by wave functions … *I say quantum, you say wave! Quantum! Wave!*
- Particles close enough for wave functions to overlap (spacing < wavelength)
- Particles are indistinguishable

Fermions: Fermi-Dirac Statistics

- Fermions obey Pauli Exclusion Principle:
  - -- Half intetger spins, only 1 particle in a quantum state

Bosons: Bose-Einstein Statistics

- Bosons *DO NOT* obey Pauli Exclusion Principle:
  - -- whole intetger spins, only 1 particle in a quantum state

Maxwell-Boltzmann Statistics: Classical Particles

Classical particles are distinguishable, only interact with each other through elastic collisions and are at a low enough density that the wave functions don't overlap.

The Maxwell-Boltzmann factor is

$$F_{MB} = Ae^{-\beta E}$$ TRex (9.27)

Maxwell-Boltzmann Speed Distribution

The speed distribution of classical particles in a gas is

$$F(v)dv = 4\pi\left(\frac{\beta m}{2\pi}\right)^{3/2} v^2 e^{-\frac{2}{2}\beta m v^2} dv$$

which can be used to determine the root-mean-square speed

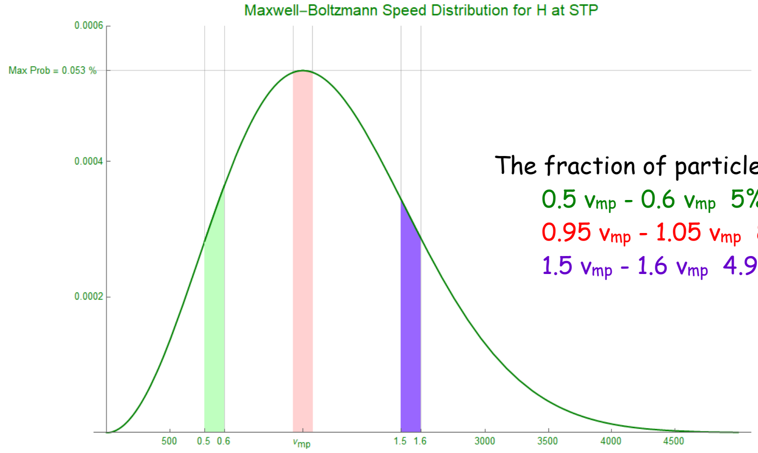
$$v_{rms}^2 = \langle v^2 \rangle = 4\pi\left(\frac{\beta m}{2\pi}\right)^{3/2} \int_0^\infty v^4 e^{-\frac{1}{2}\beta m v^2} dv = \frac{3kT}{m}$$

that is the average preferred by physicists because it gives the familiar mean kinetic energy

$$\left\langle \frac{1}{2}mv^2 \right\rangle = \frac{3}{2}kT$$

Over a small interval, an expectation value can be approximated by a product instead of an integral,

$$\langle v \rangle = \int_0^\infty vF(v)dv \approx vF(v)\Big|_{v_{min}} \Delta v = 4\pi\left(\frac{\beta m}{2\pi}\right)^{3/2} v^3 e^{-\frac{1}{2}\beta m v^2}\Big|_{v_{min}} \Delta v$$



The fraction of particles in each region is

$0.5\ v_{mp}$ - $0.6\ v_{mp}$ 5%

$0.95\ v_{mp}$ - $1.05\ v_{mp}$ 8.3%

$1.5\ v_{mp}$ - $1.6\ v_{mp}$ 4.9%

Maxwell-Boltzmann Energy Density

For an ideal gas, the density of states is the same as the number of particles, N, since a particle can have any energy, thus the number of particles with an energy between E and E + dE is

$$n_{MB}(E)dE = \frac{8\pi N}{\sqrt{2}}\left(\frac{\beta}{2\pi}\right)^{3/2} E^{1/2} e^{-\beta E} dE$$

After TRex (9.26)

so the fraction of particles with energies bewteen E and E + dE is this divided by N. The expectation value of the energy is thus

$$\langle E \rangle = \frac{8\pi}{\sqrt{2}}\left(\frac{\beta}{2\pi}\right)^{3/2} \int_0^\infty E^{3/2} e^{-\beta E} dE = \frac{3}{2}kT$$

Classical vs. Quantum Statistics

Quantum statistics must be used if the particles are dense enough that their wave functions overlap. For N particles in a volume, V at temperature T, with a de Broglie wavelength, λ = h/p, gives the criterion for using classical statistics:

$$\left(\frac{N}{V}\right)\frac{h^3}{(3mkT)^{3/2}} \ll 1$$

TRex p.336

Fermi-Dirac Statistics: Fermion Quantum Particles

Fermions have $\frac{1}{2}$-integer spins and obey the Pauli Exclusion Principal stating that only one can occupy any quantum state. The density of states for a Fermion gas is

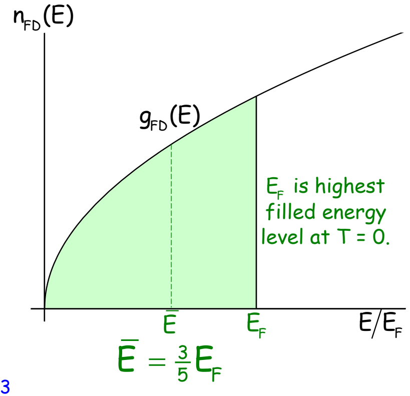
$$g_{FD}(E) = \frac{3N}{2}E_F^{-3/2}E^{1/2} \qquad \text{TRex (9.43)}$$

where the Fermi Energy, E_F is the highest occupied state at T = 0 and is given by

$$E_F = \frac{h^2}{8m}\left(\frac{3}{\pi}\frac{N}{L^3}\right)^{2/3} \qquad \text{TRex (9.42)}$$

The Fermi temperature and velocity are then

$$T_F = \frac{E_F}{k} \qquad v_F = \sqrt{\frac{2E_F}{m}} = \frac{h}{2m}\left(\frac{3}{\pi}\frac{N}{L^3}\right)^{1/3}$$



Fermi-Dirac Factor

The probability that a given state will be occupied is given by F_{FD}:

$$F_{FD} = \frac{1}{e^{(E-E_F/kT)}+1} \qquad \text{TRex (9.34)}$$

Which is equal to 1 for $E < E_F$ and zero for $E > E_F$. Since $F_{FD}(T = 393K)$ is only slightly different from $F_{FD}(T = 0) = 1$, we can work as though room temperature is equal to zero.

Fermi-Dirac Energy Density

For metals such as Silver, with 5.86×10^{28} conduction electrons per meter, the Fermi Energy is quite large, 5.503 eV (8.82×10^{-19} J) which gives a Fermi Temperature of, $kT_F = E_F \Rightarrow T_{F,\,\text{Silver}} = 63{,}750\text{K}$. Since room temperature is so much smaller than this, we can consider that we work at T = 0. Thus,

$$n_{FD}(E)dE = \frac{3N}{2}E_F^{-3/2}E^{1/2}dE \text{ at } T = 0 \qquad \text{TRex (9.44)}$$